Connections are explored between spectral descriptions of turbulence and the mean velocity profile in the equilibrium layer of wall-bounded flows using a modeled budget for the co-spectral density. Using a standard model for the wall normal velocity variance and a Rotta-like return-to-isotropy closure for the pressure-strain effects, the co-spectrum is derived. The approach establishes a relation between the von Kármán ($\kappa$), one-dimensional Kolmogorov ($C'_K$), and Rotta ($A$) constants, namely, $\kappa = (4A/7C'_K)^{-3/4}$. Depending on the choices made about small-scale intermittency corrections, the logarithmic mean velocity profile or a power-law profile with an exponent that depends on the intermittency correction are derived thereby offering a new perspective on a long standing debate.

Among the most important phenomenological theories of turbulence, two are often singled out: the von Kármán-Prandtl logarithmic law for the mean velocity profile and the Kolmogorov hypothesis (hereafter referred to as K41) for the local structure of the turbulent velocity. These two developments have often been regarded as separate given that the log-law is an outcome of restrictive boundary effects on eddy sizes responsible for mixing, while the local structure of turbulence is associated with locally homogeneous and isotropic turbulence far from any boundary. Furthermore, these two theories have stirred significant debate and continue to receive attention and novel interpretations. Intermittency corrections to the local structure of turbulence modify the $-5/3$ spectral exponent, and laws other than logarithmic have been routinely used to represent the mean velocity profile. Over the past two decades, connections between these two theories were explored via incomplete similarity and intermediate asymptotics and through the so-called “spectral link” with intriguing outcomes in both cases. A phenomenological theory that builds on these previous connections is proposed here using a co-spectral budget for the turbulent stress normal to the wall. It is shown that the co-spectral budget (i) is consistent with the onset of a $-7/3$ power-law scaling in the co-spectrum between longitudinal and normal velocity fluctuations for eddies within the inertial subrange associated with K41, (ii) leads to novel linkages among phenomenological and model closure constants, and (iii) provides a new perspective on linkages between intermittency corrections to K41 within the inertial subrange and the onset of logarithmic or power-law mean velocity profiles in the intermediate region.

For wall-bounded turbulent flows, the momentum flux or shear stress $\tau_r = -\bar{u}'w'$ plays the most central role in the dynamics. Here $u'$ and $w'$ are the turbulent longitudinal (along $x$) and vertical (wall-normal, along $z$) velocity fluctuations, and the over-line indicates appropriate averaging over coordinates of statistical homogeneity. In recent years, Gioia and co-workers have proposed an analysis of the co-spectrum and mean velocity in turbulent boundary layers.
approach that seeks to relate the detailed properties of energy spectra of isotropic turbulence to the shear stress by phenomenological arguments of near-wall eddies of particular scales. As an alternative that enables considerations to momentum transport from all turbulent scales, a co-spectral budget is considered here. The co-spectrum $F_{uw}(k)$ (where $k$ is a wavenumber or inverse eddy size) has the normalization property

$$\tau_r = -\overline{u'w'} = -\int_0^\infty F_{uw}(k) \, dk.$$  \hspace{1cm} (1)

A co-spectral budget, originally developed for locally homogeneous turbulence,\textsuperscript{9} is given as\textsuperscript{10, 11}

$$\frac{\partial F_{uw}(k)}{\partial t} + 2v k^2 F_{uw}(k) = P_{uw}(k) + T_{uw}(k) + \pi(k),$$  \hspace{1cm} (2)

where $P_{uw}(k) = \Gamma(z)E_{uw}(k)$ is the production term ($E_{uw}(k)$ the vertical velocity energy spectrum) and $\Gamma(z) = dU/dz$ is the vertical gradient of the mean velocity $U(z)$. $T_{uw}(k)$ is the co-spectral flux-transport term, and $\pi(k)$ is the velocity-pressure interaction term. It is instructive to relate these terms to the shear stress transport equation in the equilibrium boundary layer region, where the dominant terms are given by

$$\frac{\partial \overline{u'w'}}{\partial t} + 2\epsilon_{uw} = -\sigma_w^2 \Gamma(z) - \frac{\partial \overline{u'w'}}{\partial z} + R_{uw},$$  \hspace{1cm} (3)

where $\epsilon_{uw} = v \int_0^\infty k^2 F_{uw}(k) \, dk$ is the viscous dissipation, and the terms on the right-hand side represent the covariance production, turbulent transport of the covariance and pressure-velocity interactions, respectively. The latter is given as

$$R_{uw} = \frac{p'}{\rho} \left( \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \right),$$  \hspace{1cm} (4)

where $p'$ is the turbulent pressure, and $\rho$ is the fluid density. Term by term, the relations are

$$\int_0^\infty P_{uw}(k) \, dk = -\sigma_w^2 \Gamma,$$  \hspace{1cm} (5)

where $\sigma_w^2 = \overline{w'^2}$ is the vertical velocity variance ($\sigma_w^2 = \int_0^\infty E_{uw}(k) \, dk$). The co-spectral flux-transport term satisfies

$$\int_0^\infty T_{uw}(k) \, dk = \frac{\partial \overline{u'w'}}{\partial z} \approx 0,$$  \hspace{1cm} (6)

in the intermediate region as reported from wind tunnel measurements in rough- and smooth-wall boundary layers\textsuperscript{12} and from long-term near-neutral velocity measurements in the atmospheric surface layer.\textsuperscript{13} Moreover,

$$\int_0^\infty \pi(k) \, dk = \frac{p'}{\rho} \left( \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \right),$$  \hspace{1cm} (7)

which acts to de-correlate $u'$ from $w'$.\textsuperscript{14}

Considering the co-spectral transfer term $T_{uw}(k)$, it is reasonable to assume that it is negligible compared to the other terms. A plausibility argument for this statement is that its integral over $k$ equals the term $\partial \overline{w'u'w'}/\partial z$ which is known to be small as earlier noted from experiments.\textsuperscript{12, 13} With this assumption, the dominant terms that remain in stationary flows are given by

$$\Gamma(z)E_{uw}(k) + \pi(k) - 2v k^2 F_{uw}(k) \approx 0.$$  \hspace{1cm} (8)

The aim is to use this budget to establish a relation between $\Gamma(z)$, $E_{uw}(k)$ and $F_{uw}(k)$. To achieve this goal, a closure for the pressure-velocity co-spectrum $\pi(k)$ is needed. In second-order closure modeling, the integral of $\pi(k)$, $R_{uw}$, is commonly expressed as

$$R_{uw} = -C_R \overline{u'w'} + C_\Pi \sigma_w^2 \Gamma(z),$$  \hspace{1cm} (9)
where \( \tau = K/\varepsilon \) is a relaxation time scale, \( K \) is the turbulent kinetic energy, \( \varepsilon \) is the dissipation rate of \( K \), \( C_R \approx 1.8 \) is known as the Rotta constant, and \( C_2 = 3/5 \) is a constant associated with the isotropization of the production term correcting the original Rotta model\(^{14,15} \) and shown to be consistent with Rapid Distortion Theory.\(^{14} \) This model along with its associated two constants is commonly labeled as the LRR-IP for Launder-Reece-Rodi (LRR) approach with isotropization of the production (IP) term.\(^{14,16} \) This formulation is selected here because of its ability to reproduce \( R_{ww} \) computed using Direct Numerical Simulations of homogeneous shear flows.\(^{14} \) If the Rotta model is further invoked for the spectral version of this term, i.e., for \( \pi(k) \), then it follows that

\[
\pi(k) = -C_R \frac{F_{ww}(k)}{\tau(k)} + C_2 P_{ww}(k),
\]

(10)

where \( \tau(k) = \varepsilon^{-1/3}k^{-2/3} \) is now a wavenumber dependent relaxation time scale assumed\(^{10} \) to vary only with \( k \) and \( \varepsilon \) for consistency with K41. With this approximation for \( \pi(k) \), the co-spectral budget (Eq. (8)) reduces to

\[
2\nu k^2 F_{ww}(k) = (1 - C_2) \Gamma(z)E_{ww}(k) - C_R \frac{F_{ww}(k)}{\tau(k)}.
\]

(11)

The relative importance of the Rotta component and the viscous term \( 2\nu k^2 F_{ww}(k) \) is given as

\[
\frac{2\nu k^2 F_{ww}(k)}{C_R F_{ww}(k)/\tau(k)} = \frac{2}{C_R} \left( \frac{\nu^4 k^4}{\varepsilon} \right)^{1/3} \approx (k\eta)^{4/3},
\]

(12)

where \( \eta = (\nu^3/\varepsilon)^{1/4} \) is the Kolmogorov micro-scale. For \( k\eta \ll 1 \), the de-correlation due to viscous effects can be ignored relative to the Rotta term. However, as \( k\eta \gtrsim 1 \), the two terms become comparable in magnitude, but at such fine scales, \( |F_{ww}(k)| \) is sufficiently small so that ignoring contributions to \( \tau \), can be justified. Hence, this co-spectral budget equation, hereafter referred to as the equilibrium state, reduces to a balance between production and pressure-velocity interaction (i.e., destruction) terms leading to

\[
F_{ww}(k) = \frac{1}{A} \Gamma \varepsilon^{-1/3} E_{ww}(k) k^{-2/3},
\]

(13)

where \( A = C_R/(1 - C_2) \approx 4.5 \). When \( E_{ww}(k) \) is given by its K41 phenomenological form\(^2 \) \( E_{\text{Kol}}(k) = C'_{K} \varepsilon^{2/3}k^{-5/3} \), generally valid for \( \eta \ll k^{-1} \ll z \), then

\[
F_{ww}(k) = \frac{C'_{K}}{A} \Gamma \varepsilon^{1/3} k^{-2/3}.
\]

(14)

The above expression for \( F_{ww}(k) \) is in agreement with \( F_{ww}(k) = C_{ww} \Gamma \varepsilon^{1/3} k^{-7/3} \) derived from dimensional considerations\(^{10,17} \) and supported by a large corpus of measurements in high Reynolds number pipe, boundary layer and atmospheric flows,\(^{14,18,19} \) where \( C_{ww} \) is an empirical dimensionless coefficient. For one-dimensional spectra and co-spectra in which \( k \) is interpreted as one-dimensional cut along the streamwise direction \( x \), \( C_{K} = (24/55)C_{K} \), where \( C_{K} \approx 1.5 \) is the Kolmogorov constant (associated with 3-dimensional wavenumbers) one finds a \( C_{ww} = C'_{K}/A \approx 0.65/4.5 = 0.15 \). This estimate of \( C_{ww} \) is close to the accepted \( C_{ww} = 0.15 - 0.16 \) value derived from detailed co-spectral measurements interpreted as a one-dimensional cut\(^{14,20} \) along \( x \). This finding illustrates an interesting connection between \( C_{ww} \) and the (modified) Rotta model constant \( A \).

To recover a spectral link between \( E_{ww}(k) \) and \( U(z) \) similar (but not identical) to the one proposed elsewhere,\(^7 \) the mean momentum flux \( \int_{-\infty}^{\infty} F_{ww}(k) \text{d}k \) is considered within the intermediate region. The \( F_{ww}(k) \) requires description of \( E_{ww}(k) \) across all \( k \). In Figure 1, an idealized \( E_{ww}(k) \) that is constant for \( k \leq k_a \) and switches to inertial subrange (ISR) scaling for \( k \geq k_a \) is assumed, where \( k_a = z^{-1} \). The prevalence of these two regions along with \( k_a z = 1 \) is supported by data collected across many field and laboratory experiments.\(^{21-24} \) The exponential correction to the inertial range, applicable to \( k\eta \gtrsim 1 \) in the viscous range, can be considered, but for now it is not included. With
this description for $E_{w w}(k)$, $\int_{0}^{\infty} F_{w u}(k)dk$ is given as

$$\tau_t = \frac{C'_K}{A} \Gamma \epsilon^{1/3} \left( \int_{0}^{k_a} k_a^{-7/3}dk + \int_{k_a}^{\infty} k^{-7/3}dk \right),$$  \hspace{1cm} (15)$$

resulting in

$$\tau_t = \frac{C'_K}{A} \Gamma \epsilon^{1/3} \left( k_a^{-4/3} + \frac{3}{4} k_a^{-4/3} \right).$$  \hspace{1cm} (16)$$

In the equilibrium region and for the same assumptions as the momentum flux budget, the turbulent kinetic energy budget is given as

$$\frac{\partial K}{\partial t} = 0 = -u'w'\Gamma - \frac{\partial w'K}{\partial z} - \frac{1}{\rho} \frac{\partial w'p'}{\partial z} - \epsilon,$$  \hspace{1cm} (17)$$

which upon ignoring the sum of turbulent transport and pressure-velocity interaction terms results in the equilibrium state $\epsilon = \Gamma \tau_t$. Combining this estimate of $\epsilon$ with $k_a = z^{-1}$, the expression for $\tau_t$ in Eq. (16) can be re-arranged to yield

$$\Gamma(z) = \frac{\partial U}{\partial z} = \left( \frac{4A}{7C'_K} \right)^{3/4} \frac{\tau_t^{1/2}}{z}.$$  \hspace{1cm} (18)$$

Not surprisingly, if $\tau_t$ is not dependent on $z$, upon integration with respect to $z$ one obtains the log-law

$$U(z) = \left( \frac{4A}{7C'_K} \right)^{3/4} u_t \log(z) + B_o,$$  \hspace{1cm} (19)$$

where $B_o$ is an integration constant that depends on the wall roughness or on Reynolds number for smooth walls, and $u_t = \tau_t^{1/2}$ is the friction or shear velocity. The constant $(4A/(7C'_K))^{3/4} = 0.36$, which is close to $\kappa = 0.4$, confirming the links between $\kappa$, $A$, and $C'_K$. The uncertainty in the
numerical value of $A$ is certainly not below 10% so that no significance can be assigned to having obtained $\kappa = 0.36$ instead of the more commonly observed value of $\kappa = 0.4$.

The co-spectral budget offers another perspective on the ongoing debate between the use of logarithms or power-laws to describe $U(z)$ in the intermediate region.$^6$ Both power-law and logarithmic shapes for $U(z)$ can arise depending on how intermittency corrections to $E_{ww}(k)$ are formulated as shown here. Consider the influence of intermittency formulated in its most traditional form as

$$E_{ww}(k) = C_k' \kappa e^{2/3} k^{-5/3}(kL)^\alpha,$$  \hspace{1cm} (20)

where $\alpha = -\mu \eta^9$ is known as an intermittency correction and $\mu \approx 0.25$ is the usual intermittency exponent.$^8,25$–$28$ Above, $L$ is the integral length scale. In the classical theory, $C_k'$ is assumed not to vary with the Reynolds number $Re$, an assumption supported by a number of experiments reviewed elsewhere.$^{29}$ The evidence supporting the independence of $\mu$ from $Re$ could, arguably, be considered more convincing than some theories and experiments supporting the dependence of $C_k'$ and $\mu$ on $Re$.

When using $z$ as the integral length scale for the intermediate region, the resulting $E_{ww}(k) = C_k' e^{2/3} k^{-5/3}(kz)^\alpha$ for $kz > 1$. For $kz \leq 1$, inertial subrange scaling ceases to exist (for the wall-normal component). As $kz$ approaches unity, the effects of intermittency corrections on the spectrum are diminished given that intermittency buildup occurs as finer and finer scales are approached. Again using $\epsilon = \tau t^\alpha$ and the co-spectral budget, and upon ignoring the viscous and flux-transport terms as before, these assumptions lead to

$$\frac{dU}{dz} \approx \frac{(\tau t)^{1/2}}{z} \left( \frac{A}{C_k'} \frac{4}{7 - 3\alpha} \right)^{3/4}.$$

(21)

Such type of intermittency corrections alone do not alter the log-profile shape of $U(z)$ but lead to a von Kármán coefficient that very weakly depends on the intermittency exponent. Two other possibilities of the length scale used to introduce intermittency in the spectral term (i.e., $L$ in Eq. (20)) are now explored, both bounding the intermediate region. The first assumes $L = \eta$, while the other assumes $L$ is a constant $= \eta$, the height of the boundary layer (or the hydraulic radius). The $L = \eta$ was already proposed for the velocity spectrum and was shown to reproduce experiments reported in grid-generated turbulence.$^{30}$ When $L = \eta$, $E_{ww}(k) = C_k' e^{2/3} k^{-5/3}(k\eta)^\alpha_1$, where $\alpha_1$ is again an intermittency exponent. In this case, the co-spectral budget integrated over $k$ yields

$$\tau \sim \Gamma(z)^{1/3} \eta^{\alpha_1} z^{4/3 - \alpha_1}.$$

(22)

Equation (22) leads to a $dU/dz \sim z^{(12\alpha_1 - 16)/(16 - 3\alpha_1)}$, where the logarithmic law is only recovered in the limit $\alpha_1 \to 0$. Likewise, when $L = \eta$, the integrated co-spectral budget leads to

$$\frac{dU}{dz} = \left( \frac{A}{C_k'} \frac{3 - 4\alpha_2}{7 - 3\alpha_2} \right)^{3/4} \frac{\tau^{1/2}}{z} \left( \frac{z}{h} \right)^{3\alpha_2/4},$$

(23)

where $\alpha_2$ is, as before, an intermittency correction, and the outcome is again a power-law solution for $U(z)$. Evidently, intermittency corrections formulated using $L = \eta$ do not alter the log-law but yield a $\kappa$ that may weakly depend on intermittency. As is well-known, distinguishing log or power laws from mean velocity profile measurements is difficult in practice, and if such deviations exist, they would be made even more difficult to detect by the fact that $\alpha$ is on the order of 0.03 for many high Reynolds number flows. Some authors$^{31}$ have postulated the existence of a Reynolds number independent power law region for smooth wall boundary layers (sometimes called the mesolayer), which lies below the “true” logarithmic region but above the buffer layer. Our analysis, being applicable to both rough and smooth walled flows with emphasis on the region significantly above the viscous dominated region so as to ensure the development of an extensive inertial subrange, does not address the flow properties in the meso-layer.

Summarizing, links between the mean velocity and the spectrum of turbulence are gaining attention in turbulence research, as they offer analytical foresight of how self-similarity in macroscopic
variables (e.g., $U(z)$) arise from scaling laws describing the energy distribution (K41) and transport efficiencies (co-spectra) of eddies. In this work, this linkage is achieved via a simplified co-spectral budget in which the energy distribution of eddies (via $E_{ww}(k)$) naturally arises in the production term of the turbulent stress. The main closure approach applied to the velocity-pressure interaction is a revised Rotta formulation expanded to include wave-number dependencies and isotropization of the production term. With this closure, the solution to the co-spectral budget is shown to be consistent with the onset of a $-7/3$ power-law scaling in the co-spectrum. The co-spectral budget also reveals novel linkages among phenomenological (e.g., Kolmogorov constant) and closure constants (e.g., the Rotta constant).

The co-spectral approach accounts for all the wavenumber contributions to $E_{ww}(k)$ at any given $z$, a major departure from a previous approach\(^7\) that assumes only eddy sizes commensurate to $z$ efficiently transport momentum. Another important departure from this approach is that the co-spectral budget reveals that the vertical velocity spectrum, not the turbulent kinetic energy spectrum, is responsible for the vertical momentum exchange. The turbulent kinetic energy spectrum is known to exhibit different scaling laws at very low wavenumbers (e.g., von Kármán spectrum) when compared to its $E_{ww}(k)$ counterpart. Some of these differences are sometimes attributed to “inactive eddy contributions” to the longitudinal velocity spectrum $E_{ww}(k)$ absent in $E_{ww}(k)$. Even in the vicinity of $kz = 1$, many studies\(^32\) on individual velocity component spectra suggest that $E_{ww}(k)$ (the main contributor to the turbulent kinetic energy) approximately scales as $k^{-1}$ while $E_{ww}(k)$ approximately scales as $k^3$. Hence, the main velocity component contributing to the turbulent kinetic energy spectrum in the intermediate region, i.e., $E_{ww}(k)$, used in previous studies leading to the spectral link,\(^7\) significantly differs in its scaling laws from $E_{ww}(k)$. On a corollary note, it is instructive to evaluate the implications of the two-regime description for $E_{ww}(k)$ upon $\sigma_w$. The integral $\int_0^\infty E_{ww}(k)dk$ can be expanded as

$$\sigma_w^2 = C'_K \epsilon^{2/3} \left( k_0^{-5/3} \int_0^{k_0} dk + \int_{k_0}^{\infty} k^{-5/3} dk \right). \tag{24}$$

Following integration,

$$\sigma_w^2 = \frac{5}{2} C'_K \epsilon^{2/3} k_a^{2/3}. \tag{25}$$

Assuming $k_a^{-1} = z$, $\epsilon = \tau_\Gamma$, and $\Gamma(z)$ given by Eq. (18) results in $\sigma_w^2/\tau_\epsilon = \frac{5}{2} \sqrt{\frac{2}{3} A C'_K}$, or $\sigma_w/\tau_\epsilon \approx 1.7$ independent of $z$. This lack of dependence from $z$ differs from the longitudinal velocity variance $\sigma_u/\tau_\epsilon$, which was shown to scale with $z$ in the intermediate region.\(^33\) The $\sigma_w/\tau_\epsilon = 1.7$ is higher than the 1.25–1.30 value reported in pipe and boundary-layer flows.\(^14\) Any exponential cutoff at high $k$ or modulations at low $k$ due to the finite $h$ act to reduce $\sigma_w^2$ from its predicted 1.7 value.

Finally, the co-spectral budget provided a new perspective on linkages between intermittency corrections to K41 within the inertial subrange and the onset of logarithmic or power-law mean velocity profiles in the intermediate region, depending on the characteristic length-scale defining the intermittency corrections. It was shown that intermittency corrections formulated using an integral length scale $L = z$ do not alter the log-law in the intermediate region but yield a $k$ that weakly depends on the intermittency exponent. On the other hand, intermittency corrections based on the finest length scale of turbulence or a constant length scale (e.g., commensurate with the boundary layer height or pipe radius) lead to departures from the logarithmic shape and result in power-law profiles. We remark that in recent years, empirical evidence for the classic logarithmic law\(^33\) has been highlighted from very high Reynolds number laboratory studies, and so has the evidence that intermittency corrections do not disappear at arbitrarily high $Re$ (i.e., $\mu$ and $\sigma$ approach a constant) and that spectral corrections are formulated using an $L$ commensurate with the largest scale at which ISR scaling commences, which in the case of $E_{ww}(k)$ for the intermediate region, is $z$. The present results based on the co-spectrum suggest that these trends may be connected.

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