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Logarithmic scaling in the longitudinal velocity variance explained by a spectral budget

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A logarithmic scaling for the streamwise turbulent intensity $\sigma_u^2 / u_*^2 = B_1 - A_1 \ln(z/\delta)$ was reported across several high Reynolds number laboratory experiments as predicted from Townsend’s attached eddy hypothesis, where $u_*$ is the friction velocity and $z$ is the height normalized by the boundary layer thickness $\delta$. A phenomenological explanation for the origin of this log-law in the intermediate region is provided here based on a solution to a spectral budget where the production and energy transfer terms are modeled. The solution to this spectral budget predicts $A_1 = (18/55)C_\alpha/\kappa^{2/3}$ and $B_1 = (2.5)A_1$, where $C_\alpha$ and $\kappa$ are the Kolmogorov and von Kármán constants. These predictions hold when very large scale motions do not disturb the $k^{-1}$ scaling existing across all wavenumbers $1/\delta < k < 1/z$ in the streamwise turbulent velocity spectrum $E_U(k)$. Deviations from a $k^{-1}$ scaling along with their effects on $A_1$ and $B_1$ are discussed using published data and field experiments. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4837876]

I. INTRODUCTION

Scaling-laws and self-similar states remain the cornerstone of turbulence research, especially in the so-called intermediate region of wall bounded flows where production and dissipation of turbulent kinetic energy (TKE) are in balance and spectrally separated, the Kolmogorov inertial subrange theory describes the local structure of the velocity statistics, and the von Kármán-Prandtl logarithmic law describes the mean velocity ($\overline{U}$) given by

$$U^+ = \frac{1}{\kappa} \log(z^+) + A_w,$$

where $U^+ = \overline{U}/u_*$ is the normalized mean longitudinal velocity, $z$ and $z^+ = zu_*/\nu$ are, respectively, the distance and normalized distance from the wall, $u_* = \sqrt{\tau_t/\rho_f}$ is the friction (or shear) velocity, $\kappa$ is the von Kármán constant, $A_w$ is a wall constant, $\tau_t$ is the turbulent stress assumed independent of $z$ in the intermediate region, $\nu$ and $\rho_f$ are the fluid kinematic viscosity and density, respectively, and over-line designates time-averaging. The logarithmic profile shape for $U^+$ in the intermediate region has been supported by a myriad of studies, including Townsend’s attached eddy hypothesis. Besides propounding the log-law for $\overline{U}$, Townsend’s attached eddy hypothesis has also predicted a logarithmic scaling for the streamwise turbulence intensity ($\overline{u'^2}$), defined as the mean squared quantity of the streamwise turbulent velocity fluctuations of the following form:

$$\overline{u'^2} = B_1 - A_1 \log(z/\delta),$$

where $\overline{u'^2} = u'^2 / u_*^2$, the normalized longitudinal velocity variance in wall units and $\delta$ is the thickness of the turbulent boundary layer. This result has lacked robust experimental support except from a few studies until recently. Significant interest has been noticed about the topic in

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recent years following publications by Smits, McKeon, and Marusic,\textsuperscript{14} Marusic \textit{et al.},\textsuperscript{8} and Smits and Marusic\textsuperscript{15} where four different experiments at very high Reynolds numbers were compiled to illustrate the universality of Eq. (2) across a wide range of bulk Reynold’s number.\textsuperscript{3,16–19} The compass of this work is to provide a phenomenological explanation to Eq. (2) based on a spectral budget of the longitudinal velocity thereby offering another perspective on the origin of its logarithmic (or power-law) character. Specifically, a $k^{-1}$ scaling at low wavenumbers ($k$) for the streamwise turbulent velocity spectrum ($E_u(k)$) has been prevalent in turbulence literature for some time\textsuperscript{7,10,11,20–42} both for wall bounded flows and atmospheric surface layer turbulence. Those studies differ in predicting the extent of the $k^{-1}$ scaling at low $k$. One study even found a prevalent $k^{-1}$ scaling and a limited inertial subrange for $z^+ \in [220 − 890]$.\textsuperscript{43} The arguments resulting in a $k^{-1}$ scaling range from a theoretical spectral budget analysis\textsuperscript{20,21,26} to dimensional analysis\textsuperscript{32} to laboratory and field experiments.\textsuperscript{29,39} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5} Townsend’s attached eddy hypothesis has also been used in the interpretation of the aforesaid results in Refs. 53 and 54.\textsuperscript{5}

\[ \frac{\partial \bar{e}}{\partial t} + \mathbf{U}_j \frac{\partial \bar{e}}{\partial x_j} = - \overline{u'_i u'_j} \frac{\partial \mathbf{U}_i}{\partial x_j} - \frac{\partial \overline{(u'_i e)}}{\partial x_j} - \frac{1}{\rho f} \frac{\partial \overline{(u'_i P)}}{\partial x_j} - \bar{\epsilon}, \]

where $x_1 = x$, $x_2 = y$, and $x_3 = z$ are the longitudinal, lateral, and vertical directions, respectively, $U_1 = U$, $U_2 = V$, and $U_3 = W$ are the mean longitudinal, lateral, and vertical velocity components, $u'_i$ are turbulent velocity excursions around $U_i$, \( \overline{\sigma_u^2} = \frac{1}{2} \left( \sigma_u^2 + \sigma_v^2 + \sigma_w^2 \right) \) is the TKE, $\overline{\sigma_u^2} = \overline{w^2}$, and $\sigma_w^2 = \overline{w^2}$ are the root-mean-squared velocity component along directions $x_i$, respectively, and the coordinate system $x_i$ is aligned so that $x_1$ is along the $\mathbf{U}$ direction with $W = \mathbf{U} = 0$. The first and second terms on the left-hand side of Eq. (3) represent local storage and advection of $\bar{e}$ by the mean flow. On the right hand side, the first term indicates mechanical or shear production of TKE due to a finite mean velocity gradient, the second and third terms represent transport of TKE by turbulence and pressure-velocity interactions, respectively, and the last term indicates viscous dissipation of TKE. As is the case with laboratory studies, assuming stationary and planar-homogeneous flow results in Refs. 53 and 54.

\[ \frac{\partial \bar{e}}{\partial t} = 0 = -\overline{u'w'} \frac{\partial \mathbf{U}}{\partial z} = \frac{\partial}{\partial z} \left( \overline{w' e} + \overline{w' p'} \right) - \bar{\epsilon}, \]

where $\overline{u'w'} = -\bar{u}_z^2$ is the mean momentum flux. It is to be noted that $p'$ in Eq. (4) is normalized by $\rho f$. In the intermediate region of boundary-layers, the transport terms are usually small resulting in a near-balance between production and dissipation of TKE as assumed by Townsend and many others.\textsuperscript{5,54}
B. A spectral budget

If $\bar{\epsilon}$ is a conservative quantity in the turbulent energy cascade, then a simplified spectral budget representing the interplay between the same terms in Eq. (4) can be derived for any wavenumber $k$ as \[ \bar{\epsilon} = -\frac{dU}{dz} \int_{k}^{\infty} F_{ww}(p)dp + F_{TR}(k) + 2v \int_{0}^{k} \rho^{2}E_{tke}(p)dp, \] (5)

where the first, second, and third terms represent the production of TKE in the range of $[k, \infty]$, the transfer of TKE in the range $[k, \infty]$, and the viscous dissipation in the range of $[0, k]$. Two asymptotic conditions must be satisfied so that this spectral budget recovers its Reynolds-averaged TKE counterpart. The first is that at $k = 0$, $F_{TR}(0) = 0$, and

\[ \bar{\epsilon} = -\frac{dU}{dz} \int_{0}^{\infty} F_{ww}(p)dp = -\frac{dU}{dz}(\bar{u'}w'), \] (6)

so that $\int_{0}^{\infty} F_{ww}(p)dp = \bar{u'}w'$ to ensure a balance between mechanical production and $\bar{\epsilon}$ is maintained in the intermediate region. The second is that as $k \to \infty$, $F_{TR}(\infty) \to 0$, and

\[ \bar{\epsilon} \approx 2v \int_{0}^{\infty} \rho^{2}E_{tke}(p)dp, \] (7)

or $\bar{\epsilon}$ is primarily explained via the viscous term at very large $k$. The $F_{ww}(k)$ that is related to the production term, and the $F_{TR}(k)$ that is related to the action of the triple moments and pressure-velocity interactions both require closure.

In deriving closure expressions for $F_{ww}(k)$ and $F_{TR}(k)$, $E_{tke}(k)$ is related to the spectra of the individual velocity components by

\[ E_{tke}(k) = \frac{1}{2} [E_{u}(k) + E_{v}(k) + E_{w}(k)], \] (8)

and for $kz > 1$, these component-wise velocity spectra can be described by the Kolmogorov scaling (hereafter referred to as K41) given as

\[ E_{tke}(k) = C_{\alpha}k^{2/3}k^{-5/3}; E_{u}(k) = C_{K}k^{2/3}k^{-5/3}; E_{v}(k) = C_{K}k^{2/3}k^{-5/3}; E_{w}(k) = C_{K}k^{2/3}k^{-5/3}, \] (9)

where $C_{K} = (24/55)C_{K}$, $C_{K} = (18/55)C_{K}$, $C_{K} \approx 1.55$ is the Kolmogorov constant associated with three-dimensional wavenumbers, and $C_{\alpha} = (33/55)C_{K}$. Exponential (or Pao type) adjustments to these individual spectra as $k\eta \to 1$ are momentarily ignored, where $\eta = (v^{3}/\epsilon)^{1/4}$ is the Kolmogorov micro-scale, and $v$ is the kinematic viscosity of the fluid. Throughout, $k$ is interpreted as one-dimensional cut along the streamwise direction $x$ and this interpretation is adopted hereafter as invoked when converting the time domain to the wavenumber domain using Taylor’s frozen turbulence hypothesis in experiments. Last, because the TKE and the component-wise velocity spectra are known for $kz > 1$ given by their K41 scaling, it is convenient to consider the spectral budget in Eq. (5) at $k_{0} = 1/\zeta$ instead of any arbitrary $k$.

C. Modeling the production term $F_{ww}(k)$

The $F_{ww}(k)$ can be obtained via a co-spectral budget similar in form to the spectral budget above given by

\[ \frac{\partial F_{ww}(k)}{\partial t} + 2vk^{2}F_{ww}(k) = P_{ww}(k) + T_{ww}(k) + \pi(k), \] (10)
where $P_{uw}(k) = (d\overline{U}/dz)E_w(k)$ is the production term, $T_{uw}(k)$ is the co-spectral flux-transport term, and $\pi(k)$ is the velocity-pressure interaction term. Considering the co-spectral transfer term $T_{uw}(k)$, it is reasonable to assume that it may be small compared to the other terms given that its integral over $k$ is $\overline{\delta w'u'w'}/\partial z$ which is known to be minor in the intermediate region.$^{57,58}$ With these assumptions, the dominant terms in the co-spectral budget that remain in stationary flows are

$$\frac{d\overline{U}}{dz}E_w(k) + \pi(k) - 2\nu k^2 F_{uw}(k) \approx 0. \quad (11)$$

In second-order closure modeling, the integral of $\pi(k)$, $R_{uw}$, is expressed as

$$R_{uw} = -CR_1 \frac{1}{\tau} \overline{(u'u')} + C_2 \sigma_\eta^2 \frac{d\overline{U}}{dz}, \quad (12)$$

where $\tau = \overline{\tau}/\overline{\tau}$ is a relaxation time scale, $C_R \approx 1.8$ is the Rotta constant, and $C_2 = 3/5$ is a constant associated with isotropization of the production term correcting the original Rotta model$^{54,59}$ and shown to be consistent with Rapid Distortion Theory.$^{54}$ If the Rotta model is further invoked for the spectral version of this term, then it follows that$^{56}$

$$\pi(k) = -C_R \frac{F_{uw}(k)}{\tau(k)} + C_2 P_{uw}(k), \quad (13)$$

where $\tau(k) = \overline{\tau}^{-1/3}k^{-2/3}$ becomes a wavenumber dependent relaxation time scale$^{60}$ assumed to vary only with $k$ and $\overline{\tau}$ for consistency with K41. With this approximation for $\pi(k)$, the co-spectral budget reduces to$^{56}$

$$2\nu k^2 F_{uw}(k) = (1 - C_2) \frac{d\overline{U}}{dz}E_w(k) - C_R \frac{F_{uw}(k)}{\tau(k)}. \quad (14)$$

The relative importance of the Rotta component and the viscous term $2\nu k^2 F_{uw}(k)$ can be evaluated and is given as$^{56}$

$$\frac{2\nu k^2 F_{uw}(k)}{C_R F_{uw}(k)/\tau(k)} = \frac{2}{C_R} \left( \frac{\nu k^2}{\epsilon} \right)^{1/3} \approx (k\eta)^{4/3}, \quad (15)$$

Provided $k\eta \ll 1$, de-correlation between $u'$ and $w'$ due to viscous effects can be ignored relative to the Rotta term. However, it is to be noted that as $k\eta \gtrsim 1$, the two terms become comparable in magnitude, but at such fine scales, $|F_{uw}(k)|$ is sufficiently small so that ignoring contributions to the turbulent stress can be justified. With the two remaining terms describing a balance between production and pressure-velocity interaction (i.e., destruction), the co-spectral budget is now given by$^{56}$

$$F_{uw}(k) = \frac{1}{A} \frac{d\overline{U}}{dz} \overline{\tau}^{-1/3}E_w(k)k^{-2/3}, \quad (16)$$

where $A = C_R/(1 - C_2) \approx 4.5$. For $\eta \ll k^{-1} \leq z$, then

$$F_{uw}(k) = C'_R A^{-1} k^{-\tau}. \quad (17)$$

This expression agrees with $F_{uw}(k) = C_{uw}(d\overline{U}/dz)\overline{\tau}^{1/3}k^{-7/3}$ derived from dimensional considerations$^{60,61}$ and supported by a large corpus of measurements in high Reynolds number pipe, boundary layer, and atmospheric flows.$^{54,62,63}$ Also, the $C_{uw} = C'_R A \approx 0.65/4.5 = 0.15$, agrees with the accepted value.$^{54,64}$ With such a closure model for $F_{uw}(k)$,

$$\frac{d\overline{U}}{dz} \int_{k_u}^\infty F_{uw}(p)dp = -\frac{3}{4} \left( \frac{d\overline{U}}{dz} \right)^2 \frac{1}{C_{uw}} \overline{\tau}^{1/3}k_u^{-4/3}. \quad (18)$$

This completes the estimation of the production term in the spectral budget.
D. Modeling the transport term $F_{TR}(k)$

The Heisenberg model\(^{50}\) can be used to achieve a closure to $F_{TR}(k)$ and is given by

$$F_{TR}(k) = v_t(k) |\text{curl} \tilde{u}|^2 \approx 2v_t(k) \int_0^k p^2 E_{ik}\beta(p) dp,$$

where $\tilde{u}$ is a “macro-scale” component of the velocity, and $v_t(k)$ is referred to as the Heisenberg\(^{50}\) eddy viscosity. It is produced by the motion of eddies with wavenumbers greater than $k$ and is given by

$$v_t(k) = C_H \int_k^\infty \frac{E_{ik}\beta(p)}{p^3} dp,$$

where $C_H$ is the Heisenberg constant. The turbulent viscosity can be evaluated as

$$v_t(k_o) = C_H \int_k^{k_o} \frac{C_o \epsilon_{\omega}/p^{5/3}}{p^3} dp = \frac{3C_H C_o^{1/2} \epsilon_{\omega}^{1/3}}{4k_o^{2/3}},$$

so that the ratio of turbulent to molecular viscosity is given by

$$\frac{v_t(k_o)}{\nu} = \frac{3C_H C_o^{1/2}}{4(k_o\eta)^{4/3}}.$$

Because it is assumed that $k_o\eta \ll 1$, $v \ll v_t(k_o)$, and molecular effects can be ignored (as was the case in the co-spectral budget).

E. Solving the spectral budget

If $\overline{U}$ follows Eq. (1), then $d\overline{U}/dz = u_u/(k\zeta)$ and $\tilde{e} = u_u^2 d\overline{U}/dz = u_u^4/(k\zeta)$, and using $F_{uu}(k)$ from Eq. (18), $v_t$ from Eq. (21), with $v_t(k_o) \gg v$ simplifies the spectral budget in Eq. (5) to

$$\int_0^k p^2 E_{ik}\beta(p) dp = \frac{u_u^2}{\zeta^2} C_b,$$

where

$$C_b = \frac{2}{3C_H C_o^{1/2}} \frac{-(3/4) C_{uu} + \kappa^{4/3}}{\kappa^2}.$$

To solve for $E_{ik}(k)$ in Eq. (23) for $k\zeta < 1$, assume $E_{ik}(k) = a_1 k^{b_1}$ thereby reducing Eq. (23) to

$$a_1 \zeta^{-3-b_1} = \frac{u_u^2}{\zeta^2} C_b.$$

Upon using polynomial matching, $-3 - b_1 = -2$ or $b_1 = -1$, and $a_1 = 2C_b u_u^2$. This analysis suggests that $E_{ik}(k) = C_{TKE} u_u^2 k^{-1}$, thereby recovering the $-1$ power-law in the spectrum of TKE, where $C_{TKE} = 2C_b$. To link $E_{ik}(k)$ to $E_{u}(k)$, consider its definition in Eq. (8). For $k\zeta \ll 1$, $E_{u}(k) + E_{\omega}(k) \gg E_{u}(k)$ (generally, $E_{u}(k) \sim k^\beta$ for $k\zeta < 1$ as discussed elsewhere\(^{34,36}\)). Figure 1 shows measured $E_{u}(k)$, $E_{\omega}(k)$, and $E_{u}(k)$ (in regular and pre-multiplied form) computed using orthonormal wavelet transforms (OWT) providing some experimental support to the simplification $E_{u}(k)$, $E_{\omega}(k) \gg E_{u}(k)$ for $k\zeta < 1$. Also at low wavenumbers, $E_{\omega}(k)$ scales with $E_{u}(k)$ so that $E_u(k) \sim \beta E_{ik}(k) = \beta C_{TKE} u_u^2 k^{-1} = C_{TKE} u_u^2 k^{-1}$, where $\beta$ is a proportionality constant the role of which is discussed in the Appendix and $C_{TKE} = \beta C_{TKE}$. The OWT is used here (instead of Fourier spectra) because the wavelet coefficients are less sensitive to possible non-stationarity in the component-wise velocity series. These measurements were collected at 10 Hz using a triaxial sonic anemometer positioned at $z = 39.5$ m above the ground surface on a meteorological tower situated
in a 28 m tall southern hardwood forest at maximum leaf area index (LAI = 6 m² m⁻²). The figure shows three long (~6 h) and near-stationary periods collected over 3 different days (uₐ = 0.63, 0.56, 0.57 m s⁻¹) and at heights well above any viscous layer (z⁺ = 7.4, 6.6, 6.7 × 10⁵). These data sets, described elsewhere,⁴⁹ are also shown to illustrate the scaling laws of E₀(k), E₁(k), and E₆(k). The E₀(k) and E₁(k) follow the same scaling trends as assumed, which is −5/3 scaling for kző 1 and a near k⁻¹ scaling for kзо 1. On the other hand, E₆(k) follows the −5/3 scaling for kзо 1 and attains a near-uniform value for kзо < 1 consistent with numerous studies.⁵⁴,⁵⁶ Now, instead of a monotonic k⁻¹ behavior for E₆(k), it is assumed that E₆(k) attains a uniform value at k ≤ 1/H as already hinted at in Figure 1 and also depicted in Figure 2, where H = αδ, α is a non-dimensional measure of the largest eddies in the system, and δ is the boundary layer height. From continuity requirement, 

\[ E₀(k) = C_{TKE} uₐ^2 (1/H)^{-1} \text{ for } kH \leq 1. \]

To summarize,

\[
E₀(k) = \begin{cases} 
C_{TKE} uₐ^2 H^{-1}, & \text{if } kH \leq 1 \\
C_{TKE} uₐ^2 k^{-1}, & \text{if } kH > 1, kző 1 \\
C_K \zeta^{2/3} k^{-5/3}, & \text{otherwise} 
\end{cases}
\]  

(26)

The numerical value of \( C'_{TKE} \) can be inferred in multiple ways. One requires numerical estimates of \( \kappa, C₉, C_{uw}, C_H, \) and \( \beta \) as discussed in the Appendix. A simpler one assumes continuity of \( E₀(k) \) at \( k₀. \) As evidenced from Figure 1, the transition from \( k^{-5/3} \) to \( k^{-1} \) is sufficiently narrow in \( k \) to permit using this continuity constraint to determine \( C'_{TKE} \) by matching the two spectral results in Eq. (26) at \( k₀ = 1/\zeta. \) This continuity condition leads to \( C'_{TKE} = C_K' / k^{2/3}. \) Using \( \kappa = 0.4, \) and

\[
E₀(k) = \begin{cases} 
C_{TKE} uₐ^2 H^{-1}, & \text{if } kH \leq 1 \\
C_{TKE} uₐ^2 k^{-1}, & \text{if } kH > 1, kző 1 \\
C_K \zeta^{2/3} k^{-5/3}, & \text{otherwise} 
\end{cases}
\]  

(26)
FIG. 2. Schematic depicting different scaling laws of \( E_u(k) \) and \( E_w(k) \) along with their ranges. In \( E_u(k) \) (left), the solid black line depicts the \( k^{-5/3} \) scaling at \( k_z > 1 \) and the solid grey line depicts the \( k^{-1} \) scaling at \( \frac{1}{\alpha \delta} < k_z < 1 \). The dotted grey line describes the imperfect scaling in the same range \( \frac{1}{\alpha \delta} < k_z < 1 \). The dashed grey and dashed-dotted grey lines indicate the uniform \( E_u(k) \) assumed at \( k < \frac{1}{\alpha \delta} \) discussed in Subsections II G and II H. In \( E_w(k) \) (right), the solid black line depicts the \( k^{-5/3} \) scaling at \( k_z > 1 \) and the solid grey line depicts the uniform spectrum at \( k_z < 1 \).

\[
C''_K = \left( \frac{18}{55} \right) \times 1.55 = 0.55, \quad C'_{TKE} \approx 1.0. \quad \text{This value agrees with several atmospheric surface layer experiments that suggest } C'_{TKE} = 0.9 - 1.1. \quad 37, 39, 40, 65
\]

**F. The longitudinal velocity variance**

Integrating the \( E_u(k) \) from the largest scale \( H = \alpha \delta \) following the idea of Perry and Chong,\(^6\) the variance is

\[
\sigma_u^2 = \int_0^{1/H} C'_{TKE} u_*^2 H^{-1} dk + \int_{1/H}^{k_*} C'_{TKE} u_*^2 k^{-1} dk + \int_{k_*}^{\infty} C''_K \frac{\bar{\epsilon}^2}{\kappa^{2/3}} k^{-5/3} dk. \quad (27)
\]

Substituting \( \bar{\epsilon} = u_3^3 / (\kappa z) \) and performing the integration,

\[
\sigma_u^2 = C'_{TKE} u_*^2 + C'_{TKE} u_*^2 \ln \left( \frac{H}{z} \right) + \frac{3}{2} C''_K \frac{u_*^2}{\kappa^{2/3}}. \quad (28)
\]

Normalizing \( \sigma_u^2 \) by \( u_*^2 \), Eq. (29) recovers Townsend’s attached eddy hypothesis form\(^8\)

\[
\frac{\sigma_u^2}{u_*^2} = B_1 - A_1 \ln \left( \frac{z}{\delta} \right), \quad (29)
\]

where

\[
B_1 = C'_{TKE} (1 + \ln(\alpha)) + \frac{3}{2} \frac{C''_K}{\kappa^{2/3}} \quad (30)
\]

and

\[
A_1 = C'_{TKE} \quad (31)
\]

are the intercept and slope, respectively. These estimates are now compared to the range of estimates reported in Marusic et al.\(^8\) across the four very high Reynolds number experiments (\( B_1 = 1.56 - 2.3 \), and \( A_1 = 1.21 - 1.33 \)). As earlier noted, for a \( C'_{TKE} \approx 1.0 \) estimated from the continuity in \( E_u(k) \) at \( k_* \) as discussed in Sec. II E, a \( C''_K = 0.55 \) and \( \kappa = 0.4 \) result in \( A_1 = 1.01 \). If \( \alpha = 1.0 \), i.e., \( \log(\alpha) = 0 \), then \( B_1 = 2.5 \) is consistent with the upper limit reported by Marusic et al.\(^8\) These numbers are
TABLE I. The values of $\alpha$ (95% confidence level with intervals) computed using nonlinear least squares method along with the coefficient of determination ($R^2$) and the root mean square error RMSE using Eq. (29). The spectral shape associated with the model is depicted in Figure 2 using the dotted blue, solid blue, and solid and dotted black lines.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\alpha$</th>
<th>95% confidence interval</th>
<th>$R^2$</th>
<th>RMSE</th>
<th>Fitted equation</th>
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<td>(1.41, 1.72)</td>
<td>0.96</td>
<td>0.16</td>
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<td>Superpipe</td>
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<td>(0.68, 0.83)</td>
<td>0.96</td>
<td>0.19</td>
<td>Eq. (29)</td>
</tr>
<tr>
<td>LCC</td>
<td>1.36</td>
<td>(1.24, 1.48)</td>
<td>0.96</td>
<td>0.19</td>
<td>Eq. (29)</td>
</tr>
<tr>
<td>SLTEST</td>
<td>1.58</td>
<td>(0.98, 2.19)</td>
<td>0.87</td>
<td>0.37</td>
<td>Eq. (29)</td>
</tr>
</tbody>
</table>

also close to the empirical estimates of $A_1 = 1.03$ and $B_1 = 2.39$ provided in Smits, McKeon, and Marusic. The other estimates of $B_1$ and $A_1$ using the alternate estimate of $C_{TKE}$ is discussed in the Appendix. Also, it is interesting to note that for the three runs collected above the hardwood forest reported in Figure 1, $\sigma_{u’}u_*= 2.0$, $1.9$, $1.9$. For a typical daytime neutral atmospheric boundary $\delta = 1000$ m, a measurement height of about 40 m, a zero-plane displacement height of $2/3$ canopy height, $A_1 = C_{K}^*/\kappa^{2/3} = 1.01$ and a $B_1 = (1 + 3/2)A_1$, result in $\sigma_{u’}u_* = 2.4$, reasonably close to the field measurements here ($\sim 2.0$).

G. Effect of very large scale motion (VLSM)

So far, the largest length scale discussed in the problem is $a\delta$, the presence of which indicates effects of VLSM, where $a \geq 1$. The four experiments from Marusic et al. are digitized and fitted to Eq. (29) so as to determine $\alpha$. As mentioned before, $A_1$ is set at 1.01, and a nonlinear least squares method is used to fit the other parameter $\alpha$. Table I reports the values of $\alpha$ along with the coefficient of determination ($R^2$) and root mean square error (RMSE) in the first four rows. With such a three-regime spectral shape, the range of $\alpha$ (close to 1.0) indicates that the large scales contributing $\sigma_{u’}u_*$ are commensurate to the expected boundary layer height. The shape of $E_u(k)$ is schematically shown in Figure 2. The four different datasets - Melbourne, Superpipe, LCC, and SLTEST are also plotted against the predicted log-law $B_1 - A_1 \ln (\epsilon/\delta)$ in Figure 3 for three different values of $\alpha = 1.0$, $1.5$, $0.5$, where $B_1$ and $A_1$ are predicted by Eqs. (30) and (31), respectively. It is however, noted that a $k^{-1}$ regime is not observed in the Superpipe data in the $E_u(k)$ spectra as reported by Rosenberg et al. The reason for showing the predicted log-law for three different $\alpha$ is worth discussing here. It has to be noted that the Superpipe data are different from the other three datasets as the flow in case of pipe is confined (wall-bounded) in all sides, while the channel or field data are roughly bounded on one side only. The confining effect of the walls from all the sides might constrain the largest eddies in the system, and this effect can be picked up by a reduced $\alpha$, say 0.5 ($\alpha = 1.0$ would imply the largest eddy size equal to the boundary layer size). As evident from Fig. 3, a smaller $\alpha = 0.5$ places the predicted log-law close to the Superpipe data by lowering the intercept $B_1$ but keeping the slope $A_1$ undisturbed. For the sake of completeness, the log-law is also shown for a higher $\alpha = 1.5$, which would imply largest eddies larger than the boundary layer height. As expected, this places the log-law line higher up by increasing the intercept $B_1$. It is also interesting to note that the experimental data fall between these two limits of $\alpha = 0.5$ and 1.5, while the $\alpha = 1.0$ line falls in the middle of the datasets, thereby providing insight into uncertainties associated with the intercept $B_1$ mentioned in Marusic et al.

H. Imperfect scaling and deviations from a $k^{-1}$ law in the longitudinal velocity spectrum

As discussed in Sec. I, some studies have already reported a deviation from the $k^{-1}$ power-law in $E_u(k)$ at low wavenumbers $k \leq 1$ as $E_u(k) = C_{TKE} u_*^2 k^{-1} (k\delta)^\sigma$. Now assuming a uniform $E_u(k)$ at $k \leq 1/H$, the uniform value $E_u$ is derived from continuity condition at $kH = 1$ where $H = \alpha \delta$ as $C_{TKE} u_*^2 \delta (1/\alpha)^{\sigma-1}$. This shape of $E_u(k)$ is also depicted in Figure 2. In summary, the
imperfect scaling can be written as

\[
E_u(k) = \begin{cases} 
C'_{TKE} u_*^2 \delta (1/\alpha)^{p-1} & \text{if } kH \leq 1 \\
C'_{TKE} u_*^2 k^{-1}(k\delta)^{p} & \text{if } kH > 1, kz \leq 1 \\
C''_K \epsilon^{2/3} k^{-5/3} & \text{otherwise}
\end{cases}
\]  

(32)

where \( p \) can be any real number, and \( \delta \) is the height of the boundary layer as before. As before, continuity requirement at \( kz = 1 \) leads to \( C'_{TKE} = (C''_K / k^{2/3}) (z/\delta)^p \). Performing the integration as in Eq. (27) and using this value of \( C'_{TKE} \),

\[
\sigma_u^2 = \int_0^{1/H} C'_{TKE} u_*^2 \delta (1/\alpha)^{p-1} dk + \int_{1/H}^{k_1} C'_{TKE} u_*^2 \delta^{p} k^{p-1} dk + \int_{k_1}^{\infty} C''_K \epsilon^{2/3} k^{-5/3} dk.
\]

(33)

Simplifying, and normalizing by \( u_*^2 \),

\[
\frac{\sigma_u^2}{u_*^2} = \left( \frac{C''_K}{k^{2/3}} \right) \left( \frac{z}{\delta} \right)^{p} + \frac{3}{2} \frac{C''_K}{k^{2/3}} + \frac{1}{p} \frac{C''_K}{k^{2/3}} \left( 1 - \left( \frac{z}{\delta} \right)^p \frac{1}{\alpha} \right). \]

(34)

The datasets are fitted using a nonlinear least squares method with Eq. (34) and values of \( p \) and \( \alpha \) are shown in the first four rows of Table II.

From Table II, a power law deviated from \( -1 \) by \( p \) may be a plausible description to the \( E_u(k) \) scaling at \( kz \leq 1 \) for the data reported in Marusic et al.\(^8\) Also, a constant \( E_u(k) \) at \( kH \leq 1 \) seems to be a plausible representation if a near unity in \( \alpha \) is used as an evaluation metric. However, it should be emphasized that a finite \( p \) leads to power-law dependence of \( u^2 \) on \( z/\delta \) instead of logarithmic. The ranges of values of \( p \) is to be noted in Table II, indicating roughly a power-law of \( k^{-1.08} \) instead of a \( k^{-1} \) scaling. The pre-multiplied spectra presented in Rosenberg et al.\(^66\) (arguing against a \( k^{-1} \) scaling) and Nickels et al.\(^48\) (supporting a limited \( k^{-1} \) scaling), when digitized, also indicate a deviation in the pre-multiplied spectral ordinate of about 0.3, which strongly suggests the presence of a power law \( k^{-1.08} \) instead of a \( k^{-1} \) scaling.
I. The vertical velocity variance

While Townsend’s hypothesis predicts a logarithmic decrease in $\sigma_w^2/u_s^2$ with increasing $z$, it predicts a constant $\sigma_w^2/u_s^2$ in the intermediate region. A possible explanation for the $z$ independence of $\sigma_w/u_s$ is now considered. This explanation begins with the turbulent vertical velocity difference between two points separated by a distance $r$ given as

$$\Delta w(r) = w' (x + r) - w' (x)$$

whose root-mean squared value is

$$\overline{\Delta w(r)^2} = \overline{w'^2 (x + r)} + \overline{w'^2 (x)} - 2 \overline{w' (x + r) w' (x)}.$$  

(36)

For homogeneous turbulence, $\overline{w'^2 (x + r)} = \overline{w'^2 (x)}$, and this expression simplifies to

$$\overline{\Delta w(r)^2} \approx 2 \sigma_w^2 [1 - \rho(r)].$$

(37)

Here, $\rho(r)$ is the vertical velocity autocorrelation function that varies with $r$. As $r \to z$, $\rho(r) \ll 1$ and in a first order analysis, Eq. (37) suggests that $\overline{\Delta w(z)^2} \approx 2 \sigma_w^2$. Adopting inertial subrange structure function scaling for $\overline{\Delta w(z)^2} = C_K \bar{e}^{2/3} z^{2/3}$, equating $2 \sigma_w^2$ to $C_K \bar{e}^{2/3} z^{2/3}$ and noting that in the intermediate region $\bar{e} = u_s^2/\kappa z$,

$$\sigma_w^2 = \frac{C_K'}{2} \frac{u_s^2}{\kappa^{2/3}},$$

(38)

so that in normalized form,

$$\frac{\sigma_w^2}{u_s^2} = \frac{C_K'}{2} \frac{1}{\kappa^{2/3}} = B_3.$$  

(39)

Equation (39) demonstrates the $z$ independence of $\sigma_w^2$ noted in many studies. It also suggests an “upper-limit” on the constant $B_3 \approx 2.7$ when $C_K' \times 4$ and $\kappa = 0.4$. The factor 4 is needed to convert the conventional constant from a spectral to structure function. In practice, as $r \to z$, $\rho(z) \sim e^{-1}$ and $B_3$ in Eq. (39) is reduced by a factor $\sim 1 - 1/e$. This finite $\rho(z) = 1/e$ correction was estimated by assuming an exponentially decaying autocorrelation function ($\rho(r) = \exp(-rlz)$) whose integral length scale (i.e., $\int_0^\infty \rho(r)dr$) is $z$, as expected. The co-location of the integral length scale with $z$ is support by a myriad of experiments (including Fig. 1) reporting $E_u(k)$ a constant up to $kz = 1$, and then $E_u(k)$ follows its Kolmogorov scaling for $kz > 1$. The schematic in the right panel of Figure 2 depicts the expected shape of the spectrum. Hence, the peak in $k E_u(k)$, co-located with the integral length scale, is situated at $kz = 1$. The finite $\rho(z)$ adjustment leads to $B_3^{1/2} = 1.3$, which is in good agreement with numerous experiments, including those reported by Marusic et al.6 As discussed in Subsection II C, the clear separation of the spectrum at $kz = 1$ also relates to attached eddies, thereby demonstrating interconnectivity between spectral and co-spectral budgets and attached eddies.

III. CONCLUSION

A phenomenological explanation to the log-law scaling in the streamwise turbulent intensity was provided based on a solution to the spectral budget derived for the intermediate region of boundary
layers. Linking of a co-spectral budget to the spectral budget by means of a production term whose main source is $E_w(k)^{36}$ is noted. Dividing the $E_w(k)$ spectrum in two different zones at $k_a = \epsilon^{-1}$ reveals underlying connections to Townsend’s attached eddy hypothesis. The breakpoint at $k_a = \epsilon^{-1}$ is clearly observed in many $E_w(k)$ spectra within the intermediate region, dividing the eddies into attached manifesting a $k^{-5/3}$ scaling at $kz \geq 1$, and detached eddies often indicated by a flat spectrum at $kz \leq 1$, though this low-wavenumber portion is not used here beyond its consequence on $E_{\delta w}(k) \approx E_w(k)$ for this range of wavenumbers. Thus, the co-spectral budget, the spectral budget, and Townsend’s attached eddy hypothesis encoded in the shape of $E_w(k)$ are all brought under a common framework to predict the logarithmic scaling in the streamwise turbulent intensity. Because inertial subrange scaling for the $u'$ spectrum was assumed for $k > 1/z$, the coefficients associated with the logarithmic scaling in the streamwise turbulent intensity were then linked to the Kolmogorov and von Kármán constants. When the $k^{-1}$ scaling was extended to all $k < 1/z$, the predicted constants describing the log-law scaling differed by some 20% from measured values reported using laboratory experiments at high Reynolds numbers. Deviation from the $k^{-1}$ scaling in the $u'$ spectrum for $kz < 1$ have also been discussed, along with the effects of VLSM on these constants. It was demonstrated via calculations here that these spectral exponent deviations can be commensurate to $k^{-1.06}$. If so, then $\sigma^2_a/u^2$ exhibits power-law instead of logarithmic scaling in $z/\delta$, though the difference between the two in terms of statistical fitting to the data may be too small to discern.

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APPENDIX: AN ALTERNATE ESTIMATE OF $C'_{TKE}$

An alternate estimate of $C'_{TKE}$ is now discussed. To evaluate this estimate based on $C_H$, it is necessary to provide numerical values for $C'_K$, $C_{uw}$, and $C_H$ in the context of a certain interpretation of $K$. For three-dimensional isotropic turbulence, it can be shown that $C_H = (8/9)C_o^{-3/2}$ as discussed elsewhere.67 Interestingly, this isotropic estimate of $C_H$ can also be recovered from the spectral budget here when production and viscous terms are absent so that the energy transfer $F(k_a)$ at any arbitrary $k_a$ is entirely balanced by $\bar{\varepsilon}$ resulting in

$$\bar{\varepsilon} = \left[ v_1(k_a) \right] \int_0^{k_a} 2C_o(\bar{\varepsilon})^{5/3} p^{-5/3} dp = \left[ \frac{3C_H C_o^{1/2} \bar{\varepsilon}^{1/3}}{4k_{a}^{4/3}} \right] \left[ \frac{3}{4} k_{a}^{4/3} 2C_o(\bar{\varepsilon})^{2/3} \right].$$

(A1)

Noting that $\bar{\varepsilon}$ and $k_a$ cancel out in Eq. (A1), and upon re-arranging, the $C_H = (8/9)C_o^{-3/2}$ is recovered even when interpreting $k$ as a one-dimensional cut in the longitudinal direction. With this formulation for $C_H$, $C_{TKE} = 2C_H = (C_o)(3/2)(k^{4/3} - (3/4)C_{uw})/k^2$. Using $\kappa = 0.4$, $C_{uu} = 0.15$ as discussed before in the derivation of $F_{uw}(k)$, and $C_o = 1.55 \times [(18 + 24 + 24)/2]/55 = 0.93$ result in $C_{TKE} = 1.5$. Estimating $C'_{TKE}$ requires a further discussion about the proportionality constant $\beta$ because $C'_{TKE} = \beta C_{TKE}$. In the inertial subrange and at $kz > 1$, $\beta = (18/55)C_k/(33/55)C_k \approx (18/33) \approx 0.5$. However, in the integrated spectrum, there are contributions from all wavenumbers and $\beta \approx 1.0$ based on $\bar{\varepsilon} = (1/2)(\sigma^2_x + \sigma^2_y + \sigma^2_z)$ with $\sigma^2_x + \sigma^2_y \approx \sigma^2_z$. It may be surmised that $\beta$ varies between 0.5 and 1.0. With $\beta = 1.0$, $C'_{TKE} = 1.5$, thereby estimating $B_1 = C'_{TKE}(1 + \ln(\alpha)) + (3/2)C''_K/k^{2/3}$ = 3.0 and $A_1 = C'_{TKE} = 1.5$, which are higher than the numbers reported by Marusic et al.8 With a $\beta$ of 0.8 (between 0.5 and 1.0), it results in $B_1 = 2.6$ and $A_1 = 1.1$, close to the range provided by Marusic et al.8 ($B_1 = 1.56 - 2.3$ and $A_1 = 1.21 - 1.33$).